9 Functions

FUNCTIONS

A function is a special case of a relation. To be specific, let X, Y be two non-empty sets and R (or *f*) be a relation from X to Y, then R may not relate an element of X to an element of Y or it may relate an element of X to more than one element of Y. But a function relates each element of X to a unique element of Y. We have:

Definition. *If X, Y are two non-empty sets then a subset f of X × Y is called a* ***function*** *(or* ***mapping*** *or* ***map****) from X to Y iff for each $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$. It is written as $f: X \to Y$.*

Thus, a subset f of X × Y is called a function from X to Y iff

(i) for each $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$ and

(ii) no two different ordered pairs have the same first component.

In other words, a function from X to Y is a rule (or correspondence) which associates to each element x of X, a unique element y of Y.

A function '*f*' can be thought of as a mechanism (or device) which gives a unique output $f(x)$ to every input x.

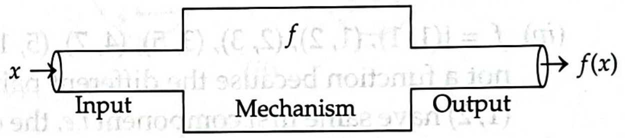


Fig. 9.1.

Image of an element. *The unique element $y \in Y$ is called the* ***image*** *of the element x of X under the function $f: X \to Y$. It is denoted by $f(x)$ i.e. $y = f(x)$. The element y is also called the* ***value*** *of the function f at x.*

Independent and dependent variables

An ***independent*** *variable is a variable that represents a quantity which is being set or manipulated in an experiment. A* ***dependent*** *variable represents a quantity whose value depends on the independent variable.*

For example: If r is radius of a circle and A is its area, then $A = \pi r^2$. Here, A depends on the value of r, so r is an independent variable and A is a dependent variable.

In the context of a function, the independent variables are the inputs to the function and the dependent variables are the outputs of the function.

For example: If y is a function of $f(x)$ *i.e.* $y = f(x)$, then x is independent variable and y is dependent variable. The value of y depends on the value of x.

Domain and range of a function

Let f be a function from X to Y, then the set X is called the ***domain*** *of the function f and the set Y is called the* ***codomain.***

The set consisting of all the images of the elements of X under the function f is called the ***range*** *of f. It is denoted by f (X). Thus,*

$$\text{range of } f = \{f(x) : \text{for all } x \in X\}.$$

Note that range of f is a subset of Y (codomain) which may or may not be equal to Y.

For example:

Let X = {1, 2, 3, 4, 5}, Y = {0, 1, 2, 3, 5, 7, 9, 11, 13} and

(*i*) f = {(1, 1), (2, 0), (3, 7), (4, 9), (5, 13)}, then f is a function from X to Y because each element of X has a unique image in Y.

Range of f = {1, 0, 7, 9, 13}.

Note that some elements of Y are not associated with any element of X.

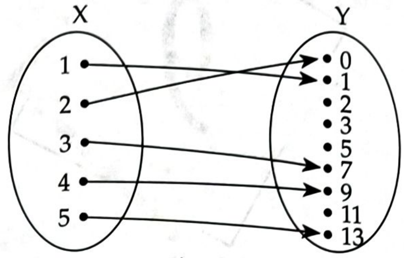


Fig. 9.2.

(*ii*) f = {(1, 3), (2, 3), (3, 5), (4, 7), (5, 5)}, then f is a function from X to Y because each element of X has a unique image in Y.

Range of f = {3, 5, 7}.

Note that the second components may repeat.

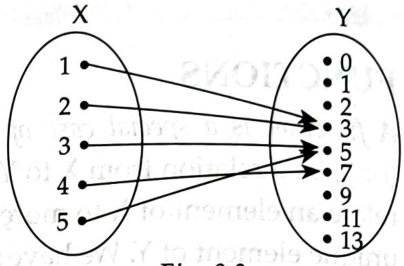


Fig. 9.3.

(*iii*) f = {(1, 5), (2, 7), (4, 9), (5, 0)}, then f is not a function from X to Y because the element 3 of X has no image in Y.

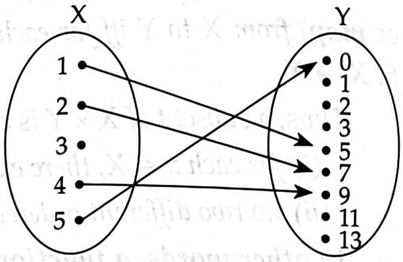


Fig. 9.4.

(*iv*) f = {(1, 1), (1, 2), (2, 3), (3, 5), (4, 7), (5, 11)}, then f is not a function because the different pairs (1, 1) and (1, 2) have same first component *i.e.* the element 1 of X has two different images.

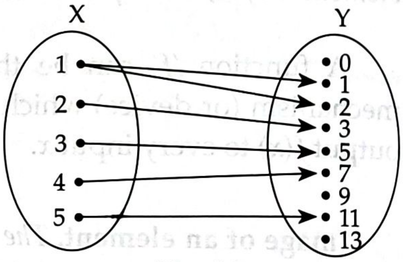


Fig. 9.5.

Main features of a function

Let f be a function from X to Y, then

(*i*) to every $x \in$ X, there exists a unique element $y \in$ Y such that $y = f(x)$.

(*ii*) no element of X can have more than one images in Y.

(*iii*) there may be elements of Y which are not associated with any element of X.

(*iv*) distinct elements of X may have same image in Y.

(*v*) function f is determined when $f(x)$ is known for all $x \in$ X.

ILLUSTRATIVE EXAMPLES

Example 1. *Which of the following relations are functions? Give reasons.*

(*i*) $R = \{(2, 1), (3, 1), (4, 2)\}$

(*ii*) $R = \{(2, 3), (\frac{1}{2}, 0), (2, 7), (-4, 6)\}$.

(*iii*) $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7)\}$.

Solution. (*i*) Domain of R = {2, 3, 4}. We note that each element of the domain of R has a unique image, therefore, the relation R is a function.

(*ii*) Domain of R = $\{2, \frac{1}{2}, -4\}$. We note that the element 2 of the domain of R has two different images 3 and 7, therefore, the relation R is not a function.

(*iii*) Domain of R = {1, 2, 3, 4, 5, 6}. We note that each element of the domain of R has a unique image, therefore, the relation R is a function.

Example 2. *If* $A = \{1, 2, 3\}$ *and f, g, h and s are relations corresponding to the subsets of* $A \times A$ *indicated against them, which of f, g, h and s are functions? In case of a function, find its domain and range.*

(*i*) $f = \{(2, 1), (3, 3)\}$ (*ii*) $g = \{(1, 2), (1, 3), (2, 3), (3, 1)\}$

(*iii*) $h = \{(1, 3), (2, 1), (3, 2)\}$ (*iv*) $s = \{(1, 2), (2, 2), (3, 1)\}$.

Solution. (*i*) *f* is not a function because the element 1 of A does not appear as the first component of ordered pairs of *f*, so 1 has no image in A.

(*ii*) *g* is not a function because the different pairs (1, 2) and (1, 3) of *g* have same first component *i.e.* the element 1 of A has two different images in A.

(*iii*) *h* is a function because each element of A has a unique image in A.

Domain of $h = \{1, 2, 3\} = A$ and range of $h = \{3, 1, 2\} = A$.

(*iv*) s is a function because each element of A has a unique image in A.

Domain of $s = \{1, 2, 3\} = A$ and range of $s = \{2, 1\}$.

Example 3. *Consider the following diagrams carefully and state whether they represent functions. Give reasons for your answer. In case of a function, write its domain and range.*

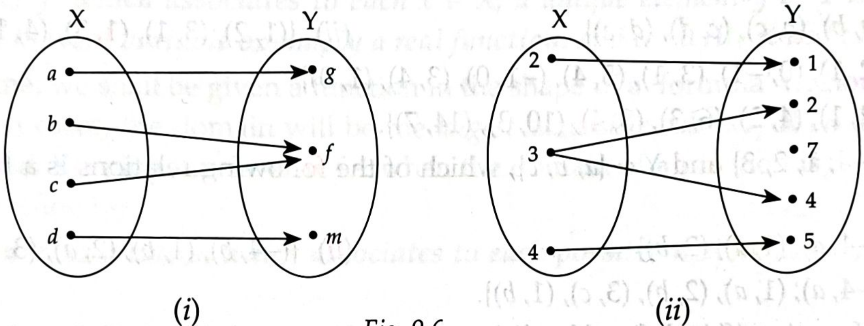


Fig. 9.6.

Solution. (*i*) The given diagram represents a function because each element of the set $\{a, b, c, d\}$ has a unique image in the set $\{g, f, m\}$.

Its domain = $\{a, b, c, d\}$ and range = $\{g, f, m\}$.

(*ii*) The given diagram does not represent a function because the element 3 of the set {2, 3, 4} has two different images 2 and 4 in the set {1, 2, 4, 5, 7}.

Example 4. *A relation 'f' is defined by* $f : x \to x^2 - 2$, *where* $x \in \{-1, -2, 0, 2\}$.

(*i*) *List the elements of f.* (*ii*) *Is f a function?*

Solution. Relation *f* is defined by $f : x \to x^2 - 2$ *i.e.* $f(x) = x^2 - 2$, where $x \in [-1, -2, 0, 2]$

(*i*) $f(-1) = (-1)^2 - 2 = 1 - 2 = -1,$

$f(-2) = (-2)^2 - 2 = 4 - 2 = 2,$

$f(0) = 0^2 - 2 = 0 - 2 = -2,$

$f(2) = 2^2 - 2 = 4 - 2 = 2.$

$\therefore \; f = \{(-1, -1), (-2, 2), (0, -2), (2, 2)\}$

(*ii*) We note that each element of the domain of *f* has a unique image, therefore, the relation *f* is a function.

Example 5. *Let* $f = \{(1, 1), (2, 3), (0, -1), (-1, -3), \ldots\}$ *be a function from* **Z** *to* **Z** *defined by* $f(x) = ax + b$, *for some integers a and b. Determine a and b.*

Solution. Given $f(x) = ax + b$.

Since $(1, 1) \in f, f(1) = 1 \Rightarrow a + b = 1$ ...(*i*)

$(2, 3) \in f, f(2) = 3 \Rightarrow 2a + b = 3$ ...(*ii*)

Subtracting (*i*) from (*ii*), we get $a = 2$.

Substituting $a = 2$ in (*i*), we get $2 + b = 1 \Rightarrow b = -1$.

Hence, $a = 2, b = -1$.

Example 6. *If A and B are finite sets such that $n(A) = p$ and $n(B) = q$, then find the number of functions from A to B.*

Solution. Any element of set A, say $x_i (1 \le i \le p)$ can be connected with an element of set B in q ways. Similarly, any other element of set A can also be connected with an element of set B in q ways. Thus, every element of set A can be connected with an element of set B in q ways. Therefore, the total number of functions from set A to set B

$= q \times q \times q \ldots p$ times $= q^p$.

Remark

Note that the number of functions from a finite set A to a finitie set B = $\{n(B)\}^{n(A)}$.

EXERCISE 9.1

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

   (*i*) $\{(a, b), (b, c), (c, d), (d, e)\}$ (*ii*) $\{(1, 2), (3, 1), (1, 3), (4, 1)\}$

   (*iii*) $\{(2, 1), (0, -1), (3, 1), (5, 4), (-1, 0), (3, 4), (1, 0)\}$

   (*iv*) $\{(2, 1), (4, 2), (6, 3), (8, 4), (10, 3), (14, 7)\}$.

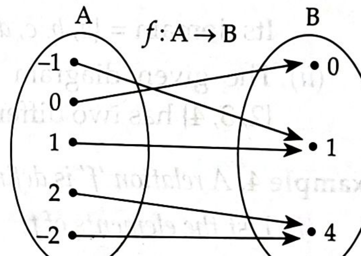
2. If $X = \{-4, 1, 2, 3\}$ and $Y = \{a, b, c\}$, which of the following relations is a function from X to Y ?

   (*i*) $\{(-4, a), (1, a), (2, b)\}$ (*ii*) $\{(-4, b), (1, b), (2, a), (3, c)\}$

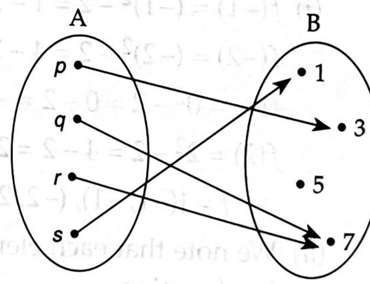
   (*iii*) $\{(-4, a), (1, a), (2, b), (3, c), (1, b)\}$.

3. (*i*) A function '*f*' is defined by $f(x) = x^2 + 1$ where $x \in \{-1, 0, 1, 3\}$. List the elements of *f*.

   (*ii*) If a function *f* is defined by $f(x) = 2x - 1$ where $x \in \{-2, 0, 3, 5\}$, then find its range.



4. The adjacent arrow diagram represents a relation. List the pairs that satisfy the relation. Is this relation a function? Also find the rule (relation) for the above correspondence.



5. Write the relation represented by the adjoining diagram, by listing the ordered pairs. State the domain, the codomain and the range of the relation. Is the relation a function?

6. $A = \{-2, -1, 1, 2\}$ and $f = \left\{\left(x, \dfrac{1}{x}\right) : x \in A\right\}$.

   (*i*) List the domain of *f*. (*ii*) List the range of *f*. (*iii*) Is *f* a function?

7. If $f(x) = ax + b$, where a and b are integers, $f(-1) = -5$ and $f(3) = 3$, find a and b.

8. If a function *f* from **R** to **R** is defined by $f = \{(x, 3x - 5) : x \in \mathbf{R}\}$, find the values of a and b given that $(a, 4)$ and $(1, b)$ belong to *f*.

Answers

1. (*i*) Yes; domain = $\{a, b, c, d\}$ and range = $\{b, c, d, e\}$
   (*ii*) No (*iii*) No
   (*iv*) Yes; domain = $\{2, 4, 6, 8, 10, 14\}$ and range = $\{1, 2, 3, 4, 7\}$
2. (*ii*)
3. (*i*) $\{(-1, 2), (0, 1), (1, 2), (3, 10)\}$ (*ii*) $\{-5, -1, 5, 9\}$
4. $\{(-1, 1), (0, 0), (1, 1), (2, 4), (-2, 4)\}$; yes; $f(x) = x^2, x \in A$
5. $\{(p, 3), (q, 7), (r, 7), (s, 1)\}$; domain = $\{p, q, r, s\}$, codomain = $\{1, 3, 5, 7\}$, range = $\{1, 3, 7\}$; yes
6. (*i*) $\{-2, -1, 1, 2\}$ (*ii*) $\left\{-\frac{1}{2}, -1, 1, \frac{1}{2}\right\}$ (*iii*) Yes
7. $a = 2, b = -3$
8. $a = 3, b = -2$

REAL FUNCTIONS

A function which has either **R** or one of its subsets as its range is called a **real valued function.** Further, if its domain is also either **R** or a subset of **R**, then it is called a **real function.** Thus, a real function is defined as:

Let R be the set of all real numbers and X, Y be two non-empty subsets of R, then a rule (or correspondence) 'f' which associates to each $x \in X$, a unique element y of Y is called a real valued function of the real variable or simply a real function, and we write it as $f : X \to Y$.

Most of the time, we shall be given a function in the shape of a 'formula' without mentioning its domain. In such cases, the domain will be the *largest admissible subset of* **R** *on which the given formula is meaningful.* Keeping this in view *i.e.* when the domain of a function is not given, a real function can be redefined as:

A real function 'f' is a rule(s) which associates to each possible real number x, a unique real number f(x).

Domain and range of a real function

Let 'f' be a real function, then:

The set of all possible real numbers x for which f(x) is a real number is called the ***domain*** *of the function f*, it is usually denoted by D_f. Thus,

$$D_f = \{x : x \in \mathbf{R}, f(x) \in \mathbf{R}\}.$$

The set of images f(x) for all $x \in D_f$ is called the ***range*** *of f, it is usually denoted by R_f. Thus,*

$$R_f = \{f(x); \text{ for all } x \in D_f\}.$$

Remarks

1. A function does not exist if its domain is empty set.
2. If the domain of a real valued function is given, then we are not to find the *admissible* subset of **R**.
3. The distinctive property of a function is that, for a given element x of the domain, $f(x)$ is uniquely determined element of codomain.

Equal functions

Two functions f and g are called ***equal****, written as $f = g$, if and only if*

(*i*) *domain of f = domain of g and*

(*ii*) $f(x) = g(x)$ *for all x in domain of f (or g).*

Thus, $f \neq g$ iff either $D_f \neq D_g$ or there exists atleast one real number $x \in D_f$ such that $f(x) \neq g(x)$.

Notes

1. If a and b are real numbers such that $a < b$, then
   (i) $(x - a)(x - b) < 0$ iff $a < x < b$ i.e. iff $x \in (a, b)$.
   (ii) $(x - a)(x - b) \leq 0$ iff $a \leq x \leq b$ i.e. iff $x \in [a, b]$.
   (iii) $(x - a)(x - b) > 0$ iff either $x < a$ or $x > b$ i.e. iff $x \in (-\infty, a) \cup (b, \infty)$.
   (iv) $(x - a)(x - b) \geq 0$ iff either $x \leq a$ or $x \geq b$ i.e. iff $x \in (-\infty, a] \cup [b, \infty)$.
2. It may be noted that $\dfrac{f(x)}{g(x)}$ has same sign as that of $f(x)\, g(x)$ because $g(x) \neq 0$ and $(g(x))^2 > 0$ for all $x \in \mathbf{R}$, and the multiplication by a positive number does not change the sign (symbol) of an inequality.
3. (i) If x is a real number, then $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$.
   (ii) $|x| \geq 0$ for all real numbers x.

ILLUSTRATIVE EXAMPLES

Example 1. *Let 'f' be a function defined by $f : x \to 5x^2 + 2, x \in \mathbf{R}$.*

(i) *Find the image of 3 under f.* (ii) *Find $f(3) \times f(2)$.*
(iii) *Find x such that $f(x) = 22$.*

Solution. Given $f(x) = 5x^2 + 2, x \in \mathbf{R}$.

(i) $f(3) = 5 \times 3^2 + 2 = 5 \times 9 + 2 = 47.$

(ii) $f(2) = 5 \times 2^2 + 2 = 5 \times 4 + 2 = 22,$
$\therefore f(3) \times f(2) = 47 \times 22 = 1034.$

(iii) $f(x) = 22 \Rightarrow 5x^2 + 2 = 22$
$\Rightarrow 5x^2 = 20 \Rightarrow x^2 = 4$
$\Rightarrow x = 2, -2.$

Example 2. *Find the domain of the following functions:*

(i) $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12}$ (ii) $f(x) = \dfrac{x + 7}{x^2 - 8x + 4}$.

Solution. (i) Given $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12}$.

For D_f, $f(x)$ must be a real number

$\Rightarrow \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12}$ must be a real number

$\Rightarrow x^2 - 8x + 12 \neq 0 \Rightarrow (x - 2)(x - 6) \neq 0$

$\Rightarrow x \neq 2, 6 \Rightarrow D_f = \mathbf{R} - \{2, 6\}.$

(ii) Given $f(x) = \dfrac{x + 7}{x^2 - 8x + 4}$.

For D_f, $f(x)$ must be a real number

$\Rightarrow \dfrac{x + 7}{x^2 - 8x + 4}$ must be a real number

$\Rightarrow x^2 - 8x + 4 \neq 0 \Rightarrow x \neq \dfrac{8 \pm \sqrt{64 - 4 \times 1 \times 4}}{2} \Rightarrow x \neq 4 \pm 2\sqrt{3}$

$\Rightarrow D_f = \mathbf{R} - \{4 \pm 2\sqrt{3}\}.$

Example 3. *If* $f(x) = \dfrac{x^2 - 3x + 1}{x - 1}$, *find* $f(-2) + f\left(\dfrac{1}{3}\right)$.

Solution. Given $f(x) = \dfrac{x^2 - 3x + 1}{x - 1}$, $D_f = \mathbf{R} - \{1\}$.

$$\therefore \quad f(-2) = \frac{(-2)^2 - 3(-2) + 1}{-2 - 1} = \frac{4 + 6 + 1}{-3} = -\frac{11}{3} \text{ and}$$

$$f\left(\frac{1}{3}\right) = \frac{\left(\frac{1}{3}\right)^2 - 3 \times \frac{1}{3} + 1}{\frac{1}{3} - 1} = \frac{\frac{1}{9} - 1 + 1}{-\frac{2}{3}} = \frac{\frac{1}{9}}{-\frac{2}{3}}$$

$$= \frac{1}{9} \times \left(-\frac{3}{2}\right) = -\frac{1}{6}.$$

$$\therefore \quad f(-2) + f\left(\frac{1}{3}\right) = -\frac{11}{3} - \frac{1}{6} = \frac{-22 - 1}{6} = -\frac{23}{6} = -3\frac{5}{6}.$$

Example 4. *Find the range of the function* $f(x) = 2 - 3x$, $x \in \mathbf{R}$, $x > 0$.

Solution. Given $f(x) = 2 - 3x$, $x \in \mathbf{R}$, $x > 0$.

For R_f, let $y = f(x) = 2 - 3x$, $x \in \mathbf{R}$, $x > 0$.

Now, $x > 0 \Rightarrow 3x > 0 \Rightarrow -3x < 0$

$\Rightarrow \quad 2 - 3x < 2 \Rightarrow y < 2$

$\Rightarrow \quad R_f = (-\infty, 2)$.

Example 5. *Find the domain and the range of the following functions:*

(*i*) $f(x) = \sqrt{x - 1}$ (*ii*) $f(x) = \dfrac{1}{\sqrt{5 - x}}$.

Solution. (*i*) Given $f(x) = \sqrt{x - 1}$.

For D_f, $f(x)$ must be a real number

$\Rightarrow \sqrt{x - 1}$ must be a real number

$\Rightarrow x - 1 \geq 0 \Rightarrow x \geq 1$

$\Rightarrow D_f = [1, \infty)$.

For R_f, let $y = f(x) = \sqrt{x - 1}$.

As $x \geq 1$, $x - 1 \geq 0$

$\Rightarrow \sqrt{x - 1} \geq 0 \Rightarrow y \geq 0$

$\Rightarrow R_f = [0, \infty)$.

(*ii*) Given $f(x) = \dfrac{1}{\sqrt{5 - x}}$.

For D_f, $f(x)$ must be a real number

$\Rightarrow \dfrac{1}{\sqrt{5 - x}}$ must be a real number

$\Rightarrow 5 - x > 0 \Rightarrow 5 > x \Rightarrow x < 5$

$\Rightarrow D_f = (-\infty, 5)$.

For R_f, let $y = \dfrac{1}{\sqrt{5 - x}}$.

As $x < 5$, $0 < 5 - x$

$\Rightarrow 5 - x > 0 \Rightarrow \sqrt{5 - x} > 0$

$\Rightarrow \quad \dfrac{1}{\sqrt{5-x}} > 0 \qquad \left(\because \dfrac{1}{a} > 0 \text{ if and only if } a > 0\right)$

$\Rightarrow \quad y > 0$

$\Rightarrow \quad R_f = (0, \infty).$

Example 6. *Find the domain and the range of the following functions:*

(*i*) $f(x) = \dfrac{x-3}{2x+1}$ (*ii*) $f(x) = \dfrac{x^2}{1+x^2}$ (*iii*) $f(x) = \dfrac{1}{1-x^2}$.

Solution. (*i*) Given $f(x) = \dfrac{x-3}{2x+1}$.

For D_f, $f(x)$ must be a real number $\Rightarrow \dfrac{x-3}{2x+1}$ must be a real number

$\Rightarrow \quad 2x+1 \neq 0 \Rightarrow x \neq -\dfrac{1}{2}$

$\Rightarrow \quad D_f =$ set of all real numbers except $-\dfrac{1}{2}$ *i.e.* $D_f = \mathbf{R} - \left\{-\dfrac{1}{2}\right\}$.

For R_f, let $y = \dfrac{x-3}{2x+1} \Rightarrow 2xy + y = x - 3$

$\Rightarrow \quad (2y-1)\,x = -y-3 \Rightarrow x = \dfrac{y+3}{1-2y}$ but $x \in \mathbf{R}$

$\Rightarrow \quad \dfrac{y+3}{1-2y}$ must be a real number $\Rightarrow 1-2y \neq 0 \Rightarrow y \neq \dfrac{1}{2}$

$\Rightarrow \quad R_f =$ set of all real numbers except $\dfrac{1}{2}$ *i.e.* $R_f = \mathbf{R} - \left\{\dfrac{1}{2}\right\}$.

(*ii*) Given $f(x) = \dfrac{x^2}{1+x^2}$.

For D_f, $f(x)$ must be a real number $\Rightarrow \dfrac{x^2}{1+x^2}$ must be a real number

$\Rightarrow \quad D_f = \mathbf{R}$ $\qquad (\because x^2 + 1 \neq 0$ for all $x \in \mathbf{R})$

For R_f, let $y = \dfrac{x^2}{1+x^2} \Rightarrow x^2y + y = x^2$

$\Rightarrow \quad (y-1)x^2 = -y \Rightarrow x^2 = -\dfrac{y}{y-1}, y \neq 1.$

But $x^2 \geq 0$ for all $x \in \mathbf{R} \Rightarrow -\dfrac{y}{y-1} \geq 0, y \neq 1$

(Multiply both sides by $(y-1)^2$, a positive real number)

$\Rightarrow \quad -y(y-1) \geq 0 \Rightarrow y(y-1) \leq 0$

$\Rightarrow \quad (y-0)(y-1) \leq 0 \Rightarrow 0 \leq y \leq 1$ but $y \neq 1$

$\Rightarrow \quad 0 \leq y < 1 \Rightarrow R_f = [0, 1).$

(*iii*) Given $f(x) = \dfrac{1}{1-x^2}$.

For D_f, $f(x)$ must be a real number $\Rightarrow \dfrac{1}{1-x^2}$ must be a real number

$\Rightarrow \quad 1 - x^2 \neq 0 \Rightarrow x \neq -1, 1$

$\Rightarrow \quad D_f =$ set of all real numbers except $-1, 1$ *i.e.* $D_f = \mathbf{R} - \{-1, 1\}$.

For R_f, let $y = \dfrac{1}{1-x^2}$

$\Rightarrow \quad 1 - x^2 = \dfrac{1}{y}, y \neq 0 \Rightarrow x^2 = 1 - \dfrac{1}{y}, y \neq 0.$

But $x^2 \geq 0$ for all $x \in D_f \Rightarrow 1 - \frac{1}{y} \geq 0$ but $y^2 > 0, y \neq 0$

(Multiply both sides by y^2, a positive real number)

$\Rightarrow y^2\left(1-\frac{1}{y}\right) \geq 0 \Rightarrow y(y-1) \geq 0 \Rightarrow (y-0)(y-1) \geq 0$

$\Rightarrow$ either $y \leq 0$ or $y \geq 1$ but $y \neq 0$

$\Rightarrow R_f = (-\infty, 0) \cup [1, \infty)$.

Example 7. *Find the domain and the range of the following functions:*

(*i*) $f(x) = \sqrt{x^2 - 4}$ (*ii*) $f(x) = \frac{1}{\sqrt{9 - x^2}}$.

Solution. (*i*) Given $f(x) = \sqrt{x^2 - 4}$.

For D_f, $f(x)$ must be a real number $\Rightarrow \sqrt{x^2 - 4}$ must be a real number

$\Rightarrow x^2 - 4 \geq 0 \Rightarrow (x + 2)(x - 2) \geq 0$

$\Rightarrow$ either $x \leq -2$ or $x \geq 2$

$\Rightarrow D_f = (-\infty, -2] \cup [2, \infty)$.

For R_f, let $y = \sqrt{x^2 - 4}$ ...(1)

As square root of a real number is always non-negative, $y \geq 0$.

On squaring (1), we get $y^2 = x^2 - 4$

$\Rightarrow x^2 = y^2 + 4$ but $x^2 \geq 0$ for all $x \in D_f$

$\Rightarrow y^2 + 4 \geq 0 \Rightarrow y^2 \geq -4$, which is true for all $y \in \mathbf{R}$. Also $y \geq 0$

$\Rightarrow R_f = [0, \infty)$.

(*ii*) Given $f(x) = \frac{1}{\sqrt{9 - x^2}}$.

For D_f, $f(x)$ must be a real number $\Rightarrow \frac{1}{\sqrt{9 - x^2}}$ must be a real number

$\Rightarrow 9 - x^2 > 0 \Rightarrow -(x^2 - 9) > 0 \Rightarrow x^2 - 9 < 0$

$\Rightarrow (x + 3)(x - 3) < 0 \Rightarrow -3 < x < 3 \Rightarrow D_f = (-3, 3)$.

For R_f, let $y = \frac{1}{\sqrt{9 - x^2}}, y \neq 0$ ...(1)

Also as the square root of a real number is always non-negative, $y > 0$.

On squaring (1), we get

$$y^2 = \frac{1}{9 - x^2} \Rightarrow 9 - x^2 = \frac{1}{y^2} \Rightarrow x^2 = 9 - \frac{1}{y^2}.$$

But $x^2 \geq 0$ for all $x \in D_f \Rightarrow 9 - \frac{1}{y^2} \geq 0$ but $y^2 > 0$

(Multiply both sides by y^2, a positive real number)

$\Rightarrow 9y^2 - 1 \geq 0 \Rightarrow y^2 - \frac{1}{9} \geq 0 \Rightarrow \left(y + \frac{1}{3}\right)\left(y - \frac{1}{3}\right) \geq 0$

$\Rightarrow$ either $y \leq -\frac{1}{3}$ or $y \geq \frac{1}{3}$ but $y > 0 \Rightarrow y \geq \frac{1}{3}$

$\Rightarrow R_f = \left[\frac{1}{3}, \infty\right)$.

Example 8. *Find the domain and the range of the following functions:*

(*i*) $f(x) = -|x|$ (*ii*) $f(x) = 1 - |x-2|$.

Solution. (*i*) Given $f(x) = -|x|$.

For D_f, $f(x)$ must be a real number

$\Rightarrow -|x|$ must be a real number, which is a real number for every $x \in \mathbf{R}$

$\Rightarrow D_f = \mathbf{R}$.

For R_f, let $y = f(x) = -|x|$.

Since $|x| \geq 0$ for all $x \in \mathbf{R} \Rightarrow -|x| \leq 0$ for all $x \in \mathbf{R}$

$\Rightarrow y \leq 0 \Rightarrow R_f = (-\infty, 0]$.

(*ii*) Given $f(x) = 1 - |x-2|$.

For D_f, $f(x)$ must be a real number

$\Rightarrow 1 - |x-2|$ must be a real number, which is a real number for every $x \in \mathbf{R}$

$\Rightarrow D_f = \mathbf{R}$.

For R_f, let $y = f(x) = 1 - |x-2|$.

Since $|x| \geq 0$ for all $x \in \mathbf{R} \Rightarrow |x-2| \geq 0$ for all $x \in \mathbf{R}$

$\Rightarrow -|x-2| \leq 0 \Rightarrow 1 - |x-2| \leq 1 \Rightarrow y \leq 1$

$\Rightarrow R_f = (-\infty, 1]$.

Example 9. *Find the domain and the range of the function f defined by* $f(x) = \dfrac{x+2}{|x+2|}$.

Solution. Given $f(x) = \dfrac{x+2}{|x+2|}$

For D_f, $f(x)$ must be a real number $\Rightarrow \dfrac{x+2}{|x+2|}$ must be a real number

$\Rightarrow \quad |x+2| \neq 0 \Rightarrow x+2 \neq 0 \Rightarrow x \neq -2$

$\Rightarrow \quad D_f =$ set of all real numbers except -2 *i.e.* $D_f = \mathbf{R} - \{-2\}$

For R_f, let $y = \dfrac{x+2}{|x+2|}$. Two cases arise.

Case I. If $x + 2 > 0$, then $|x+2| = x+2$,

$\therefore \quad y = \dfrac{x+2}{|x+2|} = 1.$

Case II. If $x + 2 < 0$, $|x+2| = -(x+2)$,

$\therefore \quad y = \dfrac{x+2}{|x+2|} = \dfrac{x+2}{-(x+2)} = -1.$

$\therefore$ On combining two cases, $R_f = \{-1, 1\}$.

Example 10. *Find the domain for which the functions* $f(x) = 2x^2 - 1$ *and* $g(x) = 1 - 3x$ *are equal.*

Solution. As the functions f and g are equal, $f(x) = g(x)$

$\Rightarrow \quad 2x^2 - 1 = 1 - 3x \Rightarrow 2x^2 + 3x - 2 = 0$

$\Rightarrow \quad 2x^2 + 4x - x - 2 = 0 \Rightarrow 2x(x+2) - 1(x+2) = 0$

$\Rightarrow \quad (x+2)(2x-1) = 0 \Rightarrow x + 2 = 0$ or $2x - 1 = 0$

$\Rightarrow \quad x = -2$ or $x = \dfrac{1}{2}$.

Hence, the domain for which the functions f and g are equal is $\left\{-2, \dfrac{1}{2}\right\}$.

EXERCISE 9.2

1. A function is defined by $f(x) = \dfrac{3x^2 + 2x - 1}{x + 1}$, $x \in \mathbf{R}, x \neq -1$. Find the value of $f(-3) + 1$.
2. A real function f is defined by $f(x) = 3x - 2$, $x \in \mathbf{R}$, find the value of $f(-3) + f(0) \times f(7)$.
3. Given $f(x) = x^3 - 1$, find x if $f(x)$ is 215.
4. A function 'f' is defined by $f(x) = x^2 + 3$, $x \in \mathbf{N}$ and $x \leq 5$.
   (*i*) Find the range of f. (*ii*) Find $f(2) \times f(5)$.
   (*iii*) Does $f(-3)$ exist? (*iv*) Find x when $f(x) = 7$.
5. Find the domain of the following real functions:
   (*i*) $f(x) = \dfrac{x^2 - 4}{x + 2}$ (*ii*) $f(x) = \dfrac{x}{x^2 + 3x + 2}$ (*iii*) $f(x) = \dfrac{x^3 - x + 3}{x^2 - 1}$.
6. Find the domain and the range of the function $f(x) = 2x^2 + 1$. Also find $f(-2)$ and the numbers which are associated with the number 51 in its range.
7. Find the domain and the range of the following functions:
   (*i*) $f(x) = \sqrt{x + 2}$ (*ii*) $f(x) = \sqrt{3 - 2x}$ (*iii*) $f(x) = \dfrac{1}{\sqrt{x - 5}}$.
8. Find the domain and the range of the following functions:
   (*i*) $f(x) = \dfrac{x^2 - 9}{x - 3}$ (*ii*) $f(x) = \dfrac{x + 1}{2x + 1}$ (*iii*) $f(x) = \dfrac{1}{4 - x^2}$.
9. Find the domain and the range of the following functions:
   (*i*) $f(x) = \sqrt{16 - x^2}$ (*ii*) $f(x) = \sqrt{x^2 - 9}$.
10. Find the domain and the range of the following functions:
    (*i*) $f(x) = |x - 3|$ (*ii*) $f(x) = 3 - |x - 2|$.
11. Find the domain and the range of the function f defined by $f(x) = \dfrac{|x - 4|}{x - 4}$.
12. Find the domain for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal.

Answers

1. -9 2. -49 3. 6
4. (*i*) $\{4, 7, 12, 19, 28\}$ (*ii*) 196 (*iii*) No (*iv*) 2
5. (*i*) $\mathbf{R} - \{-2\}$ (*ii*) $\mathbf{R} - \{-1, -2\}$ (*iii*) $\mathbf{R} - \{-1, 1\}$
6. $D_f = \mathbf{R}$, $R_f = [1, \infty)$; 9; 5, -5
7. (*i*) $[-2, \infty)$; $[0, \infty)$ (*ii*) $\left(-\infty, \dfrac{3}{2}\right]$; $[0, \infty)$ (*iii*) $(5, \infty)$; $(0, \infty)$
8. (*i*) $\mathbf{R} - \{3\}$; $\mathbf{R} - \{6\}$ (*ii*) $\mathbf{R} - \left\{-\dfrac{1}{2}\right\}$; $\mathbf{R} - \left\{\dfrac{1}{2}\right\}$ (*iii*) $\mathbf{R} - \{-2, 2\}$; $(-\infty, 0) \cup \left[\dfrac{1}{4}, \infty\right)$
9. (*i*) $[-4, 4]$; $[0, 4]$ (*ii*) $(-\infty, -3] \cup [3, \infty)$; $[0, \infty)$
10. (*i*) $\mathbf{R}$; $[0, \infty)$ (*ii*) $\mathbf{R}$; $(-\infty, 3]$
11. Domain of $f = \mathbf{R} - \{4\}$; range of $f = \{1, -1\}$ 12. $\left\{-1, \dfrac{4}{3}\right\}$